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THE COMPUTER VERSUS KEPLER*

BY OWEN GINGERICH

WE OFTEN hear, in discussions of modern high-speed computers, how an electronic machine can calculate more in a day than a man can calculate in a lifetime by older methods. It occurred to me that a concrete demonstration of some properly chosen specific case would not only be intrinsically interesting, but might shed some light on the historical situation in question, and might also provide a dramatic example of the application of computers in the history of science.

An especially appropriate example is found in the work of Kepler on the orbit of Mars, since he gives some indication of the computational time involved. In *Astronomia Nova,* Kepler describes in detail his attempt to fit a circular orbit to a series of observations of Mars at opposition. Since he wished to investigate a somewhat more general orbit than had been adopted classically, he was led to a thorny trigonometric problem that can be solved only iteratively.

Concerning this involved procedure, Kepler implores his reader: "If you are wearied by this tedious method, take pity on me, who carried out at least seventy trials of it, with the loss of much time, and don't be surprised that this already is the fifth year since I have attacked Mars, although the year 1603 was almost entirely spent on optical investigations."[1] [1].

The implication that this problem required four years must be taken with a grain of salt, but we do get a rough idea of the time involved.

It is this tedious, time-consuming procedure that I have programmed for the IBM-7094 at the Harvard Computing Center. Before describing my quite unexpected results, let me outline Kepler's problem in somewhat greater detail.

When Kepler started his investigation on the motion of Mars, in 1601, he was already a convinced Copernican, and therefore he assumed a heliostatic orbit. Nevertheless, at the beginning, he accepted the classical idea of using circles to represent the motion, and not until two years later did he work out the elliptical form of the orbit. The "vicarious orbit" that caused Kepler so much anguish and loss of time was a circle, and in the end was completely abandoned.

Kepler had in hand a dozen observations of Mars at opposition—

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[1] To this, the French astronomer Delambre replied: "Kepler was sustained by his desire to have a case against Tycho, Copernicus, Ptolemy, and all the astronomers in the world; he has tasted this satisfaction, and I don't believe he deserves our pity for making all these calculations [2].

ten from Tycho Brahe and, later, two of his own [3]. When Mars is at opposition, the sun, earth, and Mars lie in a straight line, so the heliocentric longitude of Mars is immediately known. Figure 1, reproduced from Delambre's *Histoire de l'Astronomie Moderne*, shows us the basic diagram for this problem. In the diagram, the sun is at A, and four observations of Mars, carefully chosen for a reasonably uniform distribution, are laid out from it. Note that the earth does not enter into this discussion. Now the correct elliptical orbit of Mars does not differ very

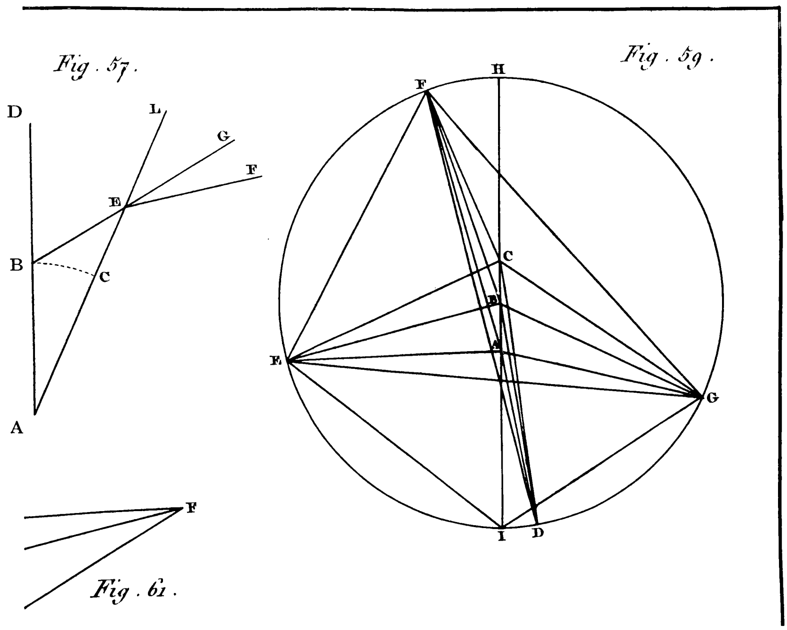

FIG. 1

much from a circle, except that the sun is at one focus and reasonably far displaced from the center. In this circular approximation, the sun lies off the center of the circle, which is at B.

We know that Mars moves most quickly when nearest the sun and slowest when at aphelion (that is, when farthest from the sun), a fact later expressed in the law of areas. Kepler believed this must be so from physical reasons, and therefore, he was already convinced that the seat of uniform angular motion in the orbit, must lie on the line through A and B, that is, on the line of apsides. In the analogous case, Ptolemy had placed this seat of uniform angular motion, or equant, equally spaced opposite A from the center of the circle. We now know that such a configuration produces the best possible approximation to an ellipse, and when we have the equant at the empty focus of the ellipse, the resulting errors in fitting the observed longitudes reach a maximum of 8′ of arc. This is the figure later found by Kepler, which, for him, proved to be such a large discrepancy from Tycho's observations that he felt obliged to abandon the circular orbits.

Kepler, however, wished to keep the spacing of A and C along the line of apsides as an unknown quantity to be determined. Also, he knew the direction of the aphelion fairly well, but he wished to improve its position. Kepler was therefore obliged to use four observations to determine all these quantities. Nowadays, we would try to use all twelve observations, combining them into a least-squares solution. This technique was, of

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$\left(\frac{AG + AE}{AG - AE}\right)$ inverted $=$ tan. 19° 37′ 40″ $\left(\frac{103021}{1544}\right)$ = tan. 18′ 11″; and AEG = 19° 55′ 51″; as also EG = $\frac{\text{AG. sin. EAG}}{\text{sin. AEG}} = \frac{52282.63271}{34088} = 97041.$

3. Since the base EG of the isosceles triangle EBG, and the vertical angle EBG are thus found, the angle BEG at the base is given, and = 25° 46′ 53″; and, therefore, BE = $\frac{\text{EG. sin. BEG}}{\text{sin. EBG}} = \frac{97041.43494}{78327} = 53860.$

4. In the triangle BEA, the angle BEA is given, for it is = BEG − AEG = 5° 51′ 2″, and also its ½ suppl. = 87° 4′ 27″. Therefore tan. ½ (BAE − ABE) = tan. ½ suppl. BEA $\left(\frac{BE - AE}{BE + AE}\right) = \frac{1957200.3121}{104599} = 58402$ = tan. 30° 17′ 8″; so that BAE = 117° 21′ 37″.

But, since in the second operation the aphelion H was found to be too far advanced in longitude, let it now, in consequence of the last correction, be considered as advanced no more than 3′ 8″, instead of 3′ 20″ beyond the longitude first assumed. Then, since AH is in 4s. 28° 47′ 8″, and AE in 8s. 26° 39′ 23″, CAE or HAE will be = 117° 52′ 5″; that is, greater by 30′ 28″ than BAE; and B is not situated in AC, but on side of it towards E. The suppositions therefore for FAH and FCH, must, one of them, or perhaps both, be false.

But these angles of anomaly cannot be varied by the mere variation of the assumed longitude of the aphelion; because no other position of it will permit the points D, F, F, G, to be situated in the circumference of the same circle; and before it can be farther varied, the mean longitudes, or the position of the lines FC, FE, &c. must be varied. This, therefore, was the next step of Kepler's procedure; and he tells us, it was not till after a great variety of unsuccessful trials, that he found his purpose would be nearly accomplished by the addition of 2′ more to the longitude of the aphelion, and of 30″ at the same time to the mean longitudes. By these additions the mean anomalies FCH, ECH, &c. are all diminished 1′ 30″ each; and we have FCH = 32° 9′ 36″; KCE = 53° 7′ 2″; KCD = 11° 0′ 44″; and KCG = 68° 18′ 1″. The angles again of equation will become AFC = 5° 8′ 26″; AEC = 9° 4′ 41″; ADC = 2° 17′ 10″; and AGC = 10° 19′ 45″; being increased 30″ in the first semi-circle of anomaly, and as much diminished in the second: consequently, the

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1CCO CCNTINUE
     N1CNT=N1CNT+1
     WRITE CUTPUT TAPE 6,101,N1CNT,N2CNT,ADDS,BCDATE,(CHS,N=1,4),BMEAN,
    WANCM,BAPP,EQN,(AF(N),N=1,4),TAN2,SUM1S,SUM2S
     GC TC K1,(220,230)
C-----ACC ARBITRARY INCREMENT IN FIRST ITERATION.
  220 ACC=RACF(0.,0.,5.,0.C1)
     ASSIGN 23C TO K1
  225 SM1=SUM1
     SM2=SUM2
     SC=SUMC
     CHCLC=CH
     CH=CH+ACC
     GC TC 190
C-----ACC PRCPORTICNAL INCREMENTS IN REMAINING ITERATIONS.
  230 IF(ACC-RACF(C.,0.,0.,10.))235,235,234
  234 ACC=ADC/(SC-SUMD)*SUMC
     IF(N1CNT-20) 2 5,300,300
C
C-----BEGIN CUTER ITERATICN.
  235 EBG= TWCPI-FAF-FAE(4) -SUM1-SUM2
     EAG=FAE(2)+FAF(3)
     AEGAGE=2.*ATANF(TNHSUPF(EAG)*ABABF(AF(4),AF(2)) )
     AEG=(PI+AFGAGF-EAG)/2.
     EG=AF(4)*SINF(EAG)/SINF(AEG)
     BEG=(PI-EPG)/2.
     BC=EG*SINF(BEG)/SINF(EBG)
     CA=1./RE
     BEA=BEG-AEG
     BAEABE=2.*ATANF(TNHSUPF(BEA)*ABABF(RE,AF(2))   )
     RAF=(PI+BAEABE-BEA)/2.
```

FIG. 2. The comparison of the Robert Small commentary with a portion of the FORTRAN program shows how closely the notations agree.

course, unavailable to Kepler. Note that the angles from A are all determined by observation. The angles from C are known relative to one another, because the motion about this point is uniform in time and the times of observations are known. The zero point of this system is to be determined, and also the direction of the aphelion AH.

Kepler starts by assuming these two quantities and solves trigonometrically the various angles of this inscribed quadrilateral. The result tells him whether or not the points lie on a circle. In the first instance they do not, so the direction AH is altered and the solution made again. A comparison of the results of these trials suggests a better position for AH, and the calculation is again repeated. This process I shall call the inner iteration. When it has finally converged, Kepler solves this triangle EGB to find if the center of circle B lies on the line CA between the sun and the equant. Again in this first instance it does not. This time, the zero point of the mean angles at C is altered, and the inner iteration is repeated. Eventually, the outer iteration also succeeds, and the points A, B, and C are found to lie on a straight line. I am sure Kepler is counting the inner iterations when he tells us that seventy trials were required.

The programming followed Kepler's procedure almost exactly. I was greatly helped by a book by Robert Small [4], which was recently reprinted through the efforts of William Stahlman. Figure 2 shows how closely the FORTRAN programming followed his notation. The principal difference in my approach is that when Kepler got close to the solution, he jumped to the answer using small corrections made by proportional parts, whereas I found it easier simply to repeat the entire calculation. Also, the program used accuracy criteria somewhat more rigid than Kepler's.

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    KEPLER'S VICARIOUS ORBIT, OR, 'THE COMPUTER VERSUS KEPLER'          12/18/63       PAGE 6

                             709/7090 FORTRAN DIAGNOSTIC PROGRAM RESULTS

        SUM1=ATANF(TAN2(4)-ATANF(TAN2(1))
03111   TOO MANY LEFT PARENTHESIS.

        SUM2= ATANF(TAN2(3)-ATANF(TAN2(2))
03111   TOO MANY LEFT PARENTHESIS.

  102   FORMAT(18H4OUTER ITERATION =I3,24X4HADM=3X,3F4.0,F5.1,8X,23HFINAL COMPARISON ANGLES/14X,4HEBG=3X,3F4.0,F5.1,
        8X,4HEAG=3X,3F4.0,F5.1, 8X,4HBAE=,3X,3F4.0,F5.1/14X,4HAEG=3X,3F4.0,F5.1,8X,4HBEG=3X,3F4.0,F5.1,8X,4HHAE=3X,3
        F4.0,F5.1/15X,3HEG=3X,F11.8,3HBE=3X,F11.8/      15X,3HBA=3X,F11.8,14X,3HCA=3X,F11.8)
04092   FORMAT STATEMENT IS INCORRECTLY WRITTEN.
                             END OF DIAGNOSTIC PROGRAM RESULTS.

    SOURCE PROGRAM ERROR.  NO COMPILATION.

EXECUTION DELETED.
```

FIG. 3. FORTRAN diagnostic. A decimal has been mispunched in place of a comma in the format statement.

After I had set up and "debugged" this program, I found that the machine could polish off the entire problem in a little less than eight seconds! This is not too surprising when we realize that only about twenty five trigonometric functions are required in each trial. Unlike Kepler, the computer does not need to look up and laboriously interpolate each of these. Instead, it computes them from scratch as needed, at the rate of 3000 per second!

At least some readers will want to know how long it took *me* to set up the program. When Kepler first arrived at Tycho's establishment, he made a bet that he would have the Mars orbit all cleaned up within eight days. When I agreed to report on this project, I too hoped to finish the calculations very quickly. But I procrastinated, and finally only eight days remained before the Christmas meeting. Thus, circumstances forced me to carry out these computations within that time span. In all, I had nine tries on the computer for this work. In the first two, the

```
N=  1   ITERATION=  1                ADD=   0. -0. -0. -0.

                         1587 MAR  6         1591 JUN  8        1593 AUG 25        1595 OCT 31
APHELION              4. 28. 44.  0.0     4. 28. 44.  0.0    4. 28. 44.  0.0    4. 28. 44.  0.0
MEAN LONGITUDE        6.  0. 47. 40.0     9.  5. 40. 19.1   11.  9. 49. 35.8    1.  7.  6. 50.3
MEAN ANOMALY          1.  2.  6. 56.0     4.  6. 56. 19.1    6. 11.  5. 35.8    8.  8. 22. 50.3
APPARENT LONGITUDE    5. 25. 43. -0.      8. 26. 39. 24.2   11. 12. 10. 31.8    1. 17. 24. 21.3
EQUATION OF CENTER    0.  5.  7. 56.0     0.  9.  4. 11.0   11. 27. 42. 20.0   11. 19. 45. 45.0
RADIUS VECTOR           5.94300860          5.07040501         4.80596131         5.23072946
TAN(HALF DIFFERENCES)  -0.07794160         -0.03456920         0.06613767         0.03087720

               SUM1   0.  6. 13. 31.0
               SUM2   0.  5. 45. 49.6

N=  2   ITERATION=  1                ADD=   0.  0.  5.  0.0

                         1587 MAR  6         1591 JUN  8        1593 AUG 25        1595 OCT 31
APHELION              4. 28. 49.  0.0     4. 28. 49.  0.0    4. 28. 49.  0.0    4. 28. 49.  0.0
MEAN LONGITUDE        6.  0. 47. 40.0     9.  5. 40. 19.1   11.  9. 49. 35.8    1.  7.  6. 50.3
MEAN ANOMALY          1.  2.  1. 56.0     4.  6. 51. 19.1    6. 11.  0. 35.8    8.  8. 17. 50.3
APPARENT LONGITUDE    5. 25. 43. -0.      8. 26. 39. 24.2   11. 12. 10. 31.8    1. 17. 24. 21.3
EQUATION OF CENTER    0.  5.  7. 56.0     0.  9.  4. 11.0   11. 27. 42. 20.0   11. 19. 45. 45.0
RADIUS VECTOR           5.92923105          5.07594460         4.77030510         5.22770876
TAN(HALF DIFFERENCES)  -0.07627323         -0.04007670         0.07149462         0.03045668

               SUM1   0.  6.  6. 22.3
               SUM2   0.  6. 23.  3.8

N=  3   ITERATION=  1                ADD=  -0.  0.  1. 52.8

                         1587 MAR  6         1591 JUN  8        1593 AUG 25        1595 OCT 31
APHELION              4. 28. 47.  7.2     4. 28. 47.  7.2    4. 28. 47.  7.2    4. 28. 47.  7.2
MEAN LONGITUDE        6.  0. 47. 40.0     9.  5. 40. 19.1   11.  9. 49. 35.8    1.  7.  6. 50.3
MEAN ANOMALY          1.  2.  3. 48.8     4.  6. 53. 12.0    6. 11.  2. 28.6    8.  8. 19. 43.1
APPARENT LONGITUDE    5. 25. 43. -0.      8. 26. 39. 24.2   11. 12. 10. 31.8    1. 17. 24. 21.3
EQUATION OF CENTER    0.  5.  7. 56.0     0.  9.  4. 11.0   11. 27. 42. 20.0   11. 19. 45. 45.0
RADIUS VECTOR           5.93441421          5.07386243         4.78371644         5.22884607
TAN(HALF DIFFERENCES)  -0.07690100          0.03800140         0.06947503         0.03061499

               SUM1   0.  6.  9.  3.6
               SUM2   0.  6.  9.  1.8

OUTER ITERATION =  1                       ADM=    0. -0. -0. -0.          FINAL COMPARISON ANGLES
          ELG=    4.  8. 26. 51.6          EAG=    4. 20. 44. 57.1         BAE=     3. 27. 34. 19.1
          AEG=    0. 19. 55. 57.8          HEG=    0. 25. 46. 34.2         HAE=     3. 27. 52. 17.0
           EG=   5.70434964                 HE=   5.38831323
           BA=   0.11485431                 CA=   0.18558683
```

Fig. 4. Intermediate computer printout. The inner iterations are carried out until SUM1 and SUM2 agree; in the outer iteration angles BAE and HAE are compared, and the process repeated until they match.

computer system detected errors of typography and nomenclature, so those trials "went up in smoke," as Kepler might say[2] (see Figure 3).

This was followed by a series of runs in which other logical flaws were detected—for example, there turned out to be an error in the Robert Small book, which I had blindly followed. By the sixth try, I already had in hand one very interesting result, after a total of eight minutes of computer time. In the ensuing runs, I corrected several more errors and also computed with different initial conditions, as I shall explain. Altogether, I used 12.4 minutes of IBM-7094 time. Now that the program has been written and "debugged," additional cases require only the eight seconds quoted above. Figure 4 illustrates an example of the output.

[2] "Itaque causae Physicae cap XLV in fumos abeunt." [4a]

The results I have just quoted sound more like a publicity release for electronic computers than a serious paper in the history of science. However, one quite remarkable fact turned up in this investigation. Instead of requiring seventy trials as Kepler did, the computer program, using identical methods, took only nine trials! In fact, we might have anticipated this result without doing any calculations at all, from the following considerations. Suppose the aphelion and the zero point of the mean longitudes are originally known to 1° (actually they were much

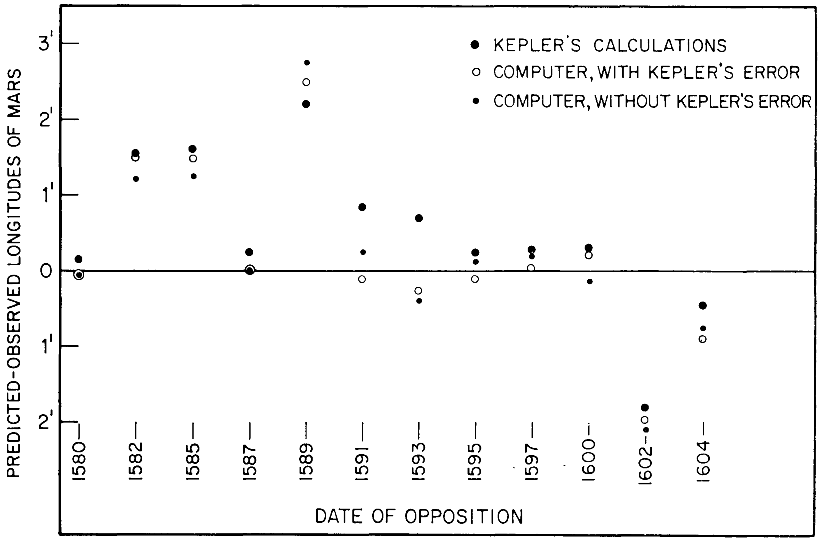

FIG. 5

better known than this). Suppose we wish to get these to 30″ of arc, that is, an improvement by a factor of 120. Since 2^7 is 128, 7 inner iterations should be required in each of 7 outer iterations, if the error is halved each time. This total number of iterations, about 50, should probably be halved because the inner and outer iterations are not independent, and as the outer iteration converges, the inner set will require fewer than 7 tries each time. Furthermore, since the problem turns out to be fairly linear, we can use proportional parts to speed the convergence, and hence we might again halve the number of iterations, making about 12. On the other hand, we make an initial try, then a try with an arbitary displacement, and finally a try with proportional parts based on the first results. Thus, three tries in each inner iteration, and three outer iterations, give a minimum of nine trials by this method, precisely the number used by the computer.

Why, then, did Kepler require seventy trials? Since Kepler already started with an arbitrary correction to Tycho's zero point on the mean longitudes, we suspect that he may have used many trials to reach the starting point shown in *Astronomia Nova*. Therefore, the calculations were repeated, starting directly from Tycho's figures. Now, thirteen iterations are required, still a very small number.

I can only conclude that Kepler was horribly plagued by numerical errors, that his trials accidentally diverged nearly as often as they converged. No wonder he was so frustrated in his attempt to solve this problem, which was apparently just at the limit of his computational ability! Do we have any evidence for this conclusion? Yes. At the very

Fig. 6. Kepler's original manuscripts, including the 900 pages of Mars calculations, are still preserved in Leningrad. "Deo et Publico" is the motto of Catherine the Great, who purchased the volumes for the Russian Academy of Science in 1773. Photograph courtesy Phillips Library, Harvard College Observatory. [10]

beginning of his calculation, Kepler makes numerical errors in three of his eight starting angles—errors of the same order of magnitude as the corrections he was seeking. These errors were noted both by Small and by Delambre. I therefore programmed the computer to solve the problem both with and without this initial error. The final solution appears comparatively insensitive to these errors, but it is curious to note that Kepler gets about the same answer *with* the errors that the machine computes *without*!

After Kepler completed his solution with four of the twelve oppositions, he carefully calculated the predicted positions for all twelve observations [5]. The results, shown in Figure 5, exhibit several interesting features.

First of all, since the solution was carried out exactly for the oppositions of 1587, 1591, 1593, and 1595, the same observed positions ought to be predicted by the theory. But here, Kepler has taken a very curious step: he corrects each of the positions for the advance of nodes of Mars—a curious step because the correction is made *after* the main calculation instead of *before* [6]! Thus, only the pivotal 1587 opposition must predict exactly the observed position; yet, as the graph indicates, Kepler has made a small computational error of 15″. Given a uniform motion of the nodes, the 1591, 1593, and 1595 observations should show increasing errors, yet again this is not the case. Compared to the machine calculations, Kepler's results for 1591 and 1593 show computational errors as large as 1′. One final comment: note from the graph how Kepler's errors generally increase the deviations between observation and prediction, *except* for the most discordant cases!

The best possible solution with this type of model, as stated previously, leaves errors up to 8′ of arc. We see here that Kepler was incredibly lucky in his particular choice of observations—or perhaps we should say unlucky, because, with larger errors, he would probably have recognized the inadequacy of this construction earlier. As a test, I chose other well-distributed sets of four oppositions as the basis of the solution, and I indeed found larger errors, up to 8′ of arc.

I hope this study has shed some light on the difficulties encountered by Kepler, and perhaps on his computational ability. My thesis, that his calculations were incredibly loaded with numerical errors, has already been observed in another section of *Astronomia Nova* by O. Neugebauer [7]. Perhaps it will someday be further confirmed by a full analysis of the 900 pages of original manuscript computations, still extant in Leningrad [8]. I do not wish, however, to detract in any way from the magnitude of Kepler's scientific achievement. Perhaps the most appropriate conclusion would be a further quotation from *Astronomia Nova:*

"There will be some clever geometers such as Vieta who will think it is something great to demonstrate the inelegance of this method. (As a matter of fact, Vieta has already made this charge against Ptolemy, Copernicus, and Regiomontanus.) Well, let them go solve this scheme themselves by geometry, and they will for me be a great Apollo. For me it suffices to draw four or five conclusions from one argument (in which there are included four observations and two hypotheses), and to have shown by the light of geometry an inelegant thread for finding the way out of the labyrinth. If this method is difficult to grasp, how much more difficult it is to investigate things without any method" [9].

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